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DISTRIBUTION OF A SUM OF WEIGHTED CHI-SQUARE VARIABLES

BY

H. SOLOMON and M. A. STEPHENS

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TECHNICAL REPORT NO. 5
JUNE 2, 1977

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20. ~~In~~ this paper ~~we~~ discuss how to obtain percentage points of the distribution of a sum of k weighted chi-square variables. A table of points is given for selected weights, for $k = 6, 8, 10$ (existing tables are for $k \leq 5$). The new points have been calculated by a technique of Imhof ~~(1961)~~ which gives very accurate values but is expensive in computer time. Therefore two new approximations are considered: a Pearson curve fit using four moments and a χ^2 fit using three. Both give excellent results in the upper tail, the tail usually needed for practical applications; the χ^2 fit appears superior to other approximations in the lower tail.

chi-square

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Distribution of a Sum of Weighted Chi-Square Variables

1. INTRODUCTION

Distributions of quadratic forms of the type

$$Q_k = \sum_{j=1}^{k} c_j (x_j + a_j)^2 \tag{1}$$

where x_j are i.i.d. standard normal variables (i.e., mean 0 and variance 1), and where c_j and a_j are non-negative constants, arise in many problems in statistics. A brief survey of such problems, with references, is in Jensen and Solomon (1972). We employ their notation in this report. An important class of problems, where there has been much recent activity, concerns asymptotic theory of goodness of fit tests. Statistics related to Pearson's chi-square, when unknown parameters are estimated by various methods, and when cells are allowed to be data-dependent, have asymptotic distributions of the Q_k type; for a survey see Moore (1976). Statistics based on the empirical distribution function (EDF statistics) also have asymptotic distributions of this type; references are given below.

Exact significance points of Q_k, for selected c_k, and all $a_k = 0$, have been published, for $k = 2$ and 3, by Grad and Solomon (1955) and by Solomon((1960), reproduced in Owen (1962)), and for $k = 4$ and 5, by Johnson and Kotz (1968); Solomon gives, also, tables of $P_k(t) = P(Q_k < t)$, for certain values of t. The mathematical difficulties involved in obtaining exact values increase with k, and several approximate methods have been suggested. The most accurate, as we show later, appears to be that of Imhof (1961), in which the characteristic function of the Q_k distribution is inverted numerically to give $P_k(t)$ for given t. In the calculations, an integral with an infinite upper limit must be calculated for each t. Imhof gives a bound on the accuracy obtained when this upper

limit is replaced by a finite T. This is apart from inaccuracies in the method of numerical integration. Thus a measure of accuracy can be attained. The Imhof method has been used by Durbin and Knott (1972), Durbin, Knott and Taylor (1975) and Pettitt and Stephens (1975) in approximating distributions of EDF statistics. These require k to be infinite and make use of a modification of Imhof's technique suggested by Durbin and Knott.

2. NEW SIGNIFICANCE POINTS FOR Q_k

It is a convenience to have exact points at hand whenever possible, so for selected c_k, and all $a_k = 0$, we give significance points of Q_k, for $k = 6, 8, 10$, in Table 1. These have been calculated by Imhof's method. They will also provide useful anchor points for comparison with other approximations already in the literature or to be suggested in Section 3. In Table 2, among several comparisons, we give exact values of significance points, and values calculated by Imhof's method to a high order of accuracy. It can be seen that the Imhof technique gives excellent results.

3. NEW APPROXIMATIONS FOR Q_k

There is clearly a need for an approximation to the Q_k distribution for problems where a_k do not appear in existing Tables; such would be the case for example, in the distribution theory of statistics of the chi-square type referred to above. The Imhof method can be almost regarded as exact, but it does not adapt easily to step-by-step increases in t

such as are needed for good inverse interpolation to obtain significance points, and when high accuracy is sought, the authors have found it relatively expensive in computer time. The need for another good approximation therefore exists. Several authors have searched for such approximations and comparisons and references are given in Jensen and Solomon (1972). Jensen and Solomon also give an approximation of the Wilson-Hilferty type which appears to be one of the most effective. Their approximation takes $z = (Q_k/\theta_1)^h$, where θ_1 is the mean of Q_k, and approximates z by a normal distribution where the mean and variance of the approximating distribution depend on the first three cumulants of Q_k.

We now propose two additional approximations: (1) fitting a Pearson curve with the same first four moments as Q_k; and (2) fitting $Q_k = Aw^r$, where w has the χ_p^2 distribution, and where A, r, p are determined by the first three moments of Q_k.

Previous authors (Patnaik (1949), Pearson (1959)) have fitted Pearson curves using less than four moments. Considerable experience with four-moment Pearson curve fitting suggests this method will be very accurate in the long upper tail of Q_k (the tail required for most applications). A practical drawback is that, once the moments have been calculated, interpolation is necessary, either in the tables of Johnson, Nixon, Amos and Pearson (1963), also reproduced in <u>Biometrika Tables for Statisticians</u>, Volume 2 (Pearson and Hartley (1972)), or in extensions of these tables recently produced by Amos and Daniel (1971) and by Bouwer and Bargmann (1973).

The Pearson curve fits will be much less effective in the lower tail;

matching the moments does not even insure that the value of $P_k(0) = 0$, i.e., that the distribution of Q_k "starts" at zero. This weakness is shared by other approximations, including that of Jensen and Solomon. However, approximation (2) above does automatically start at zero. These considerations suggest it will be better as an approximation, at least in the lower tail.

3.1 The three-moment chi-square fit

The distribution of Q_k is to be fitted by $Q_k = Aw^r$, where w has the χ_p^2 distribution and the constants A, p, r will be found by matching moments. Let $p/2 = v$, and let $C = \Gamma(v)$; the moments of z are then

$$\mu = E(z) = A\, 2^r\{\Gamma(r+v)\}/C ,$$

$$\mu_2' = A^2\, 4^r\{\Gamma(2r+v)\}/C ,$$

$$\mu_3' = A^3\, 8^r\{\Gamma(3r+v)\}/C .$$

Define

$$R_2 = \mu_2'/\mu^2 = C\ \Gamma(2r+v)/\{\Gamma(r+v)\}^2$$

and

$$R_3 = \mu_3'/\mu^3 = C^2\ \Gamma(3r+v)/\{\Gamma(r+v)\}^2 .$$

Given R_2 and R_3, these equations can be solved for r and v and then A is obtained from the expression for μ. Computer routines are available to perform these operations and then to calculate probabilities or significance points of χ^2, even with non-integer degrees of freedom. Significance points for χ^2 with degrees of freedom differing by 0.2

are given in Pearson and Hartley (1972).

The χ^2 approximation is not sensitive to employing for p the closest integer to p. If R_2 is then used to solve for r, a good approximation will often result. Further details on this method of approximation, with other areas of application, will be given in a later report.

3.2 Accuracy of various approximations

Table 2 compares significance points for several distributions Q_k for which exact (E) values exist in Solomon (1960) or Johnson and Kotz (1968); the approximations are Imhof's (I), the Jensen-Solomon approximation (J), and the new approximations, the four-moment Pearson curve fit (P) and the new three-moment χ^2 fit (S). In Table 3 these comparisons are continued for higher values of k, for which no exact values exist. In these tables, the values of c_k have been chosen to sum either to 1 or to k, to enable comparisons to be made directly with values given in the references. Jensen and Solomon compare values of $P_k(k)$ rather than significance points, so in Table 4 we give some comparisons of this type. In Tables 2, 3, and 4, values of $\sqrt{\beta_1}$ and β_2 are given in square brackets beneath the values of c_k, where $\beta_1 = \mu_3^2/\mu_2^3$ is a measure of skewness and $\beta_2 = \mu_4/\mu_2^2$ is a measure of kurtosis.

3.3 Comments

In comparing approximations with exact values, the important quantity is α', the exact significance level realized by an approximate point calculated for significance level α. The values in Table 2 show immediately the excellent accuracy of the Imhof technique, and values in Table 1 obtained by Imhof's method, and quoted again in Table 3, may for practical

purposes be regarded as accurate. Apart from this, all three approximations (P, J, S) generally give excellent accuracy in the upper tail. In the lower tail, the Pearson curves often are very poor, especially for higher values of $\sqrt{\beta_1}$ and β_2 ; Tables 2 and 3 show the clear supremacy of S over P . Table 4 suggests that S is more accurate than J in both tails; however, all approximations (except Imhof's) become relatively less good in the lower tail as the skewness and kurtosis of Q_k increase. Jensen and Solomon have already demonstrated an overall supremacy of J over other approximations discussed by them. From the picture presented here S is as good as or better than J and this merits consideration as an approximation if only three moments are to be used and especially for accuracy in the lower tail. Probability levels for S are easily obtained all along the curve and this is useful in combining significanec tests; one of the occasions when the lower tail becomes important.

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TABLE 1

Percentage Points for Q_k — All a_j are Zero

	c-values	α									
		.01	.025	.05	.10	.25	.75	.90	.95	.975	.99
k = 6	.2 .2 .2 .2 .1 .1	.1354	.1974	.2634	.3559	.5613	1.3077	1.8001	2.1473	2.4834	2.9135
	.3 .2 .2 .1 .1 .1	.1335	.19	.2522	.3424	.5448	1.3046	1.8309	2.2140	2.5922	3.0899
	.3 .3 .2 .1 .05 .05	.1162	.1670	.2242	.3090	.5073	1.3157	.19023	2.3326	2.7573	3.3140
	.3 .3 .1 .1 .1 .1	.1240	.1814	.2429	.3299	.5277	1.3025	1.8651	2.2828	2.6984	3.2513
	.4 .3 .15 .05 .05 .05	.1044	.1506	.2032	.2821	.4722	1.3145	1.9695	2.4641	2.9610	3.6224
	.4 .25 .15 .1 .05 .05	.1127	.1619	.2174	.2997	.4931	1.3064	1.9277	2.3998	2.8776	3.5195
	.4 .2 .2 .1 .05 .05	.1139	.1636	.2196	.3027	.4973	1.3065	1.9178	2.3816	2.8523	3.4875
	.4 .2 .1 .1 .1 .1	.1215	.1778	.2381	.3235	.5178	1.2923	1.8787	2.3315	2.7988	3.4289
	.5 .3 .05 .05 .05 .05	.0906	.1310	.1774	.2481	.4243	1.3126	2.0655	2.6489	3.2436	4.0451
	.5 .2 .1 .1 .05 .05	.1053	.1512	.2032	.2804	.4637	1.2883	1.9851	2.5440	3.1264	3.9231
	.5 .1 .1 .1 .1 .1	.1126	.1649	.2211	.3010	.4850	1.2768	1.9455	2.5000	3.0857	3.8832
	.6 .1 .1 .1 .05 .05	.0937	.1383	.1868	.2570	.4265	1.2625	2.0729	2.7493	3.4534	4.4221
k = 8	.2 .2 .2 .2 .05 .05 .05 .05	.1717	.2306	.2930	.3814	.5763	1.2933	1.2933	2.12	2.4521	2.8799
	.2 .2 .2 .1 .1 .1 .05 .05	.1842	.2460	.3108	.4014	.5966	1.2844	1.7386	2.0614	2.3745	2.7794
	.2 .2 .1 .1 .1 .1 .1 .1	.1970	.2618	.3288	.4212	.6168	1.2771	1.7007	1.996	2.2895	2.6649
	.3 .3 .1 .1 .05 .05 .05 .05	.1597	.2147	.2731	.3563	.5428	1.2862	1.8424	2.2592	2.6753	3.2252
	.3 .2 .2 .1 .05 .05 .05 .05	.1656	.2225	.2829	.3688	.5596	1.2891	1.8075	2.1878	2.5646	3.0613
	.3 .2 .1 .1 .1 .1 .05 .05	.1779	.2378	.3006	.3887	.5801	1.2785	1.7674	2.1298	2.4930	2.9777
	.4 .3 .05 .05 .05 .05 .05 .05	.1404	.1894	.2420	.3182	.4950	1.2912	1.9381	2.4313	2.9279	3.5896
	.4 .2 .1 .1 .05 .05 .05 .05	.1572	.2111	.2685	.3502	.5333	1.2751	1.8563	2.3096	2.7764	3.4116
	.5 .2 .05 .05 .05 .05 .05 .05	.1368	.1843	.2353	.3090	.4795	1.2700	1.9659	2.5263	3.1099	3.9082
	.5 .1 .1 .1 .05 .05 .05 .05	.1477	.1983	.2521	.3286	.5012	1.2510	1.9239	2.4827	3.0693	3.8717
k = 10	.2 .2 .2 .1 .05 .05 .05 .05 .05 .05	.2138	.2763	.3409	.4284	.6125	1.2691	1.7122	2.0291	2.3387	2.7415
	.2 .2 .1 .1 .1 .1 .05 .05 .05 .05	.2266	.2915	.3583	.4480	.6331	1.2619	1.6716	1.9649	2.2506	2.6192
	.2 .1 .1 .1 .1 .1 .1 .1 .05 .05	.2399	.3070	.3758	.4675	.6535	1.2571	1.6340	1.8975	2.1518	2.4857
	.3 .25 .1 .05 .05 .05 .05 .05 .05 .05	.2009	.2588	.3183	.4007	.5794	1.2656	1.7768	2.1611	2.5461	3.0563
	.3 .2 .1 .1 .05 .05 .05 .05 .05 .05	.2075	.2682	.3309	.4161	.5963	1.2620	1.7412	2.1013	2.4630	2.9503
	.3 .15 .1 .1 .1 .05 .05 .05 .05 .05	.2172	.2788	.3417	.4276	.6096	1.2571	1.7132	2.0565	2.4056	2.[illegible]
	.4 .2 .05 .05 .05 .05 .05 .05 .05 .05	.1889	.2434	.2998	.3780	.5500	1.2565	1.8328	2.2866	2.7543	3.3909
	.4 .1 .1 .1 .05 .05 .05 .05 .05 .05	.1987	.2566	.3166	.3980	.5710	1.2429	1.7860	2.2279	2.6296	3.3357
	.5 .1 .05 .05 .05 .05 .05 .05 .05 .05	.1773	.2294	.2836	.3578	.5187	1.2290	1.9027	2.4648	3.0512	3.8526

TABLE 2

Comparison of Exact (E) Percentage Points and Approximations I, P, S, J (see Section 3)

c-values [$\sqrt{\beta_1}, \beta_2$]		α							
		0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
0.5, 0.4, 0.1 [1.974, 9.000]	E	----	----	.098	.164	2.188	2.820	-----	-----
	I	.032	.060	.097	.164	2.187	2.818	3.450	4.290
	P	.061	.079	.109	.166	2.191	2.818	3.447	4.284
	S	.027	.053	.091	.158	2.184	2.809	3.441	4.275
0.5, 0.3, 0.2 [1.932, 9.000]	E	----	----	.110	.183	2.122	2.708	-----	-----
	I	.036	.068	.110	.183	2.122	2.708	3.302	4.102
	P	.056	.082	.119	.187	2.127	2.712	3.302	4.093
	S	.040	.071	.114	.186	2.125	2.713	3.308	4.103
	J	.029	.060	.103	.175	2.132	2.740	3.365	4.217
1.5, 1.5, 0.5, 0.5 [1.771, 7.920]	E	.261	.425	.627	.947	8.120	10.203	12.283	15.033
	I	.259	.424	.627	.947	8.120	10.202	12.282	15.032
	P	.387	.510	.673	.953	8.141	10.205	12.261	14.984
	S	.252	.415	.614	.933	8.115	10.177	12.247	14.990
	J	.212	.373	.574	.894	8.152	10.287	12.456	15.385
2.5, 0.5, 0.5, 0.5 [2.444, 12.612]	E	.226	.369	.545	.826	8.540	11.343	14.281	18.297
	I	.225	.368	.545	.826	8.540	11.342	14.279	18.294
	P	.723	----	.807	.939	8.661	11.403	------	18.133
	S	.202	.331	.495	.765	8.564	11.232	14.051	17.980
	J	.180	.295	.440	.683	8.707	11.766	15.141	20.094
1.8, 1.8, 0.6, 0.4, 0.4 [1.773, 7.957]	E	.448	.677	.943	1.347	9.921	12.419	14.914	18.214
	I	.449	.677	.943	1.347	9.921	12.419	14.913	18.213
	P	.667	.813	1.011	1.351	9.951	12.419	14.879	18.143
	S	.415	.640	.904	1.372	9.902	12.358	14.837	18.130
	J	.370	.589	.850	1.253	9.983	12.567	15.208	18.801
3.0, 0.5, 0.5, 0.5, 0.5 [2.460, 12.750]	E	.403	.609	.848	1.211	10.407	13.778	17.307	22.259
	I	.404	.609	.848	1.211	10.407	13.778	17.308	22.127
	P	1.104	----	1.201	1.355	10.562	13.847	------	21.919
	S	.344	.528	.750	1.103	10.428	13.591	16.958	21.735
	J	.324	.485	.680	.992	10.645	14.285	18.568	24.813

TABLE 3

Comparison of Approximate Percentage Points

c-values $[\sqrt{\beta_1}, \beta_2]$		α							
		.01	.025	.05	.10	.90	.95	.975	.99
0.3, 0.2, 0.1, 0.1, 0.05 (6 times)	I	.208	.268	.331	.416	1.741	2.101	2.463	2.950
[1.593, 7.380]	P	.244	.289	.340	.415	1.747	2.102	2.457	2.932
	S	.213	.270	.330	.413	1.741	2.096	2.454	2.937
	J	.206	.261	.318	.398	1.771	2.159	2.558	3.107
0.4, 0.1, 0.1, 0.1, 0.05 (6 times)	I	.199	.257	.317	.398	1.786	2.228	2.630	3.336
[2.065, 10.406]	P	----	----	----	----	-----	-----	-----	-----
	S	.204	.256	.312	.389	1.794	2.214	2.655	3.274
	J	.200	.247	.296	.367	1.849	2.363	2.936	3.794
0.2, 0.2, 0.2, 0.2, 0.1, 0.1	I	.135	.197	.263	.356	1.800	2.147	2.483	2.914
[1.259, 5.444]	P	.144	.201	.263	.355	1.801	2.148	2.482	2.912
	S	.139	.198	.263	.355	1.801	2.148	2.484	2.912
	J	.131	.192	.258	.353	1.801	2.151	2.491	2.932
0.4, 0.2, 0.1, 0.1, 0.1, 0.1	I	.122	.178	.238	.324	1.879	2.332	2.799	3.429
[1.828, 8.750]	P	.172	.209	.255	.327	1.892	2.340	2.794	3.410
	S	.135	.186	.242	.324	1.885	2.335	2.794	3.420
	J	.128	.176	.229	.308	1.917	2.412	2.928	3.648

TABLE 4

Comparison of Values of $P_k(t) = P(Q_k < t)$

c-values $[\sqrt{\beta_1}, \beta_2]$		t: 0.1	0.2	0.8	1.0	2.2	3.0	5.8
0.4, 0.3, 0.3 [1.683, 7.339]	E	.0405	.1047	.5086	.6102	.9134	.9698	.9896
	J	.0449	.1065	.5077	.6105	.9143	.9697	.9893
	S	.0397	.1039	.5091	.6107	.9134	.9697	.9896
0.5, 0.4, 0.1 [1.974, 9.000]	E	.0517	.1282	.5363	.6282	.9013	.9590	.9828
	J	.0644	.1395	.5371	.6289	.9019	.9586	.9820
	S	.0566	.1328	.5335	.6257	.9019	.9595	.9831

c-values $[\sqrt{\beta_1}, \beta_2]$		t: 0.5	1.0	3.0	3.6	8.5	11.0	14.0
2.5, 0.7, 0.4, 0.4 [2.427, 12.475]	E	.0448	.1385	.5197	.6005	.8980	.9452	.9731
	J	.0645	.1742	.5415	.6153	.8944	.9407	.9686
	S	.0534	.1514	.5153	.5935	.8975	.9465	.9745
2.0, 1.0, 0.7, 0.3 [2.011, 9.647]	E	.0375	.1189	.4862	.5728	.9078	.9568	.9821
	J	.0439	.1320	.5001	.5833	.9046	.9531	.9792
	S	.0373	.1198	.4864	.5721	.9075	.9569	.9823